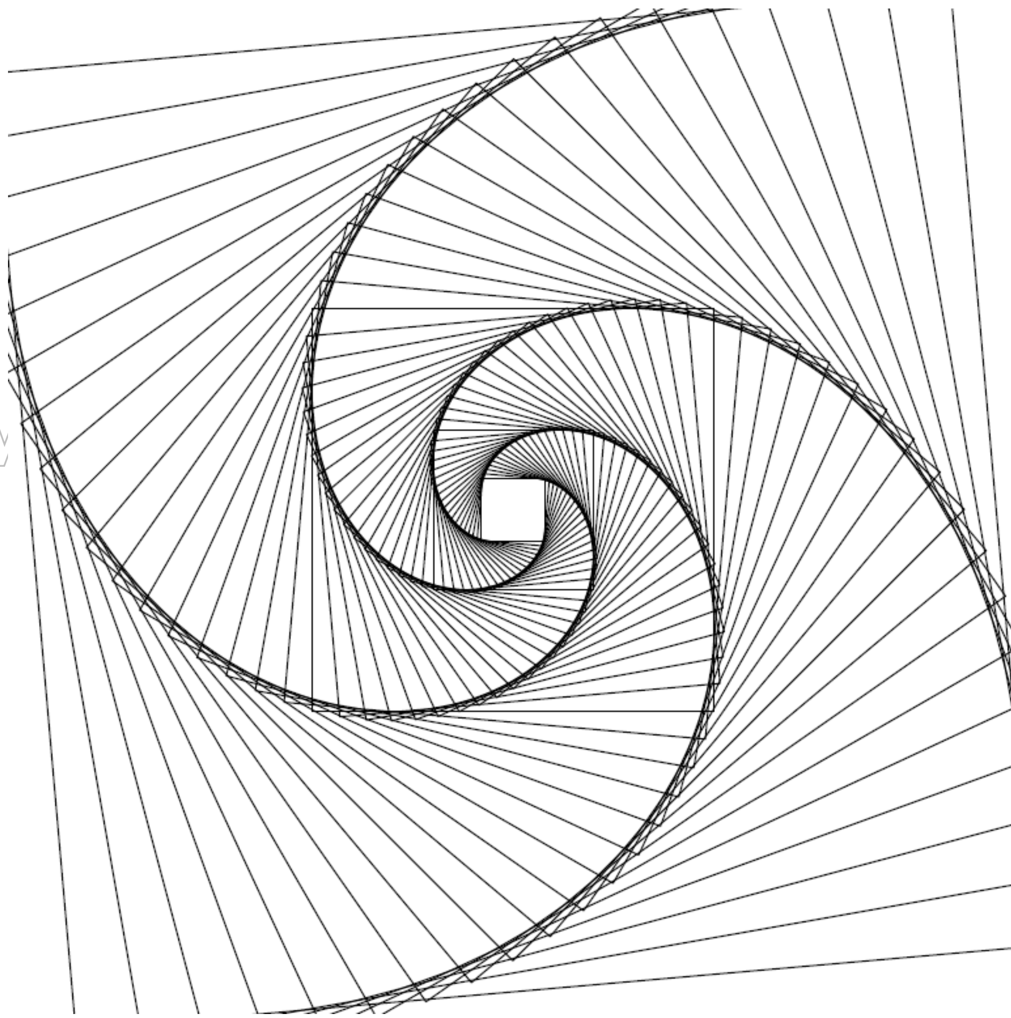
DR. ALVIN'S PUBLICATIONS

DIFFERENTIATION

DR. ALVIN ANG



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PART I

DERIVATIVES OF COMMON FUNCTIONS

A. COMMON FUNCTIONS

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x \qquad \frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$$

Source 1 (Khin 2019)

B. RULES OF DIFFERENTIATION

Product Rule: $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain Rule: $\frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx}$

Source 2 (Khin 2019)

C. INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

Source 3 (Khin 2019)

D. LOG & EXPONENTIAL FUNCTIONS

$$\frac{d}{dx}a^{f(x)} = \ln a\left(a^{f(x)}\right)f'(x)$$

$$\frac{d}{dx}\log_a f(x) = \frac{f'(x)}{f(x)\ln a}$$

Source 4 (Khin 2019)

PART II

APPLICATIONS OF DIFFERENTIATION

A. IMPLICIT VS PARTIAL VS PARAMETRIC DIFFERENTIATION

Implicit Differentiation

- Only involves (x,y)

$$x^2 + 6x - 8y + 5y^2 = 13$$

$$2x + 6 - 8\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$2x + 6 = (8 - 10y)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x+6}{8-10y} = \frac{x+3}{4-5y}$$

Partial Differentiation

- Involves (x,y,z)

$$z = \ln(x\sin y)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x}$$

and

$$\frac{\partial z}{\partial y} = \frac{\cos y}{\sin y} = \cot y$$

Parametric Differentiation

- Involves t

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$x = t^2 \Rightarrow \frac{dx}{dt} = 2t$ and $y = 2t \Rightarrow \frac{dy}{dt} = 2$

Then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}$.

B. TANGENTS AND NORMAL

$$\frac{dy}{dx}\text{(tangent) or } -\frac{1}{\frac{dy}{dx}}\text{(normal)}$$

Source 5 (Khin 2019)

C. MAXIMA AND MINIMA

FINDING TANGENTS

Example

Given the curve $x^2 - 6xy + \frac{1}{y} = 0$, $y \neq 0$, find $\frac{dy}{dx}$ in terms of x and y. Hence, find the equation of the tangent to the curve which is parallel to the x-axis.

Solution

$$x^2 - 6xy + \frac{1}{y} = 0$$

Differentiating wrt x, $2x - 6y - 6x\frac{dy}{dx} - \frac{1}{y^2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{2x - 6y}{6x + \frac{1}{y^2}}$$

$$= \frac{y^2(2x - 6y)}{6xy^2 + 1}.$$

Implicit Differentiation

Tangent // x-axis : $\frac{dy}{dx} = 0 \Rightarrow y = 0$ (N.A.) or $x = 3y$

Linear Equation that connects all gradient = 0 on the curve

Sub $x = 3y$ into $x^2 - 6xy + \frac{1}{y} = 0$, we have

$(3y)^2 - 6(3y)y + \frac{1}{y} = 0$

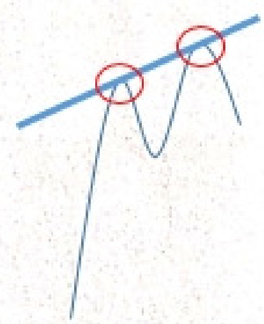
$\Rightarrow -9y^2 + \frac{1}{y} = 0$

$\Rightarrow -9y^3 + 1 = 0$

$y^3 = \frac{1}{9}$.

$\therefore$ equation of tangent is $y = \frac{1}{\sqrt[3]{9}}$.

Intersection of Line and Curve.



Source 6 (Khin 2019)

Example

The equation of a curve is $x^2 - 2xy + 2y^2 = 1$. Find the equation of the normal which is parallel to the line $y = x$.

Solution

$x^2 - 2xy + 2y^2 = 1$

Differentiate wrt x:

$$2x - 2x\frac{dy}{dx} - 2y + 4y\frac{dy}{dx} = 0$$

$$(2x - 4y)\frac{dy}{dx} = 2x - 2y$$

$$\frac{dy}{dx} = \frac{x - y}{x - 2y}$$

Implicit Differentiation

Gradient function of normal = $-\frac{x-2y}{x-y}$

$$-\frac{x-2y}{x-y} = 1$$

$$\Rightarrow -x + 2y = x - y$$

$$3y = 2x$$

$\therefore y = \frac{2}{3}x$. (Since the normal is parallel to the line $y = x$ which has gradient of 1)

Linear Equation that connects all Gradient of Normal = 1 on the curve

Sub $y = \frac{2}{3}x$ into the curve $x^2 - 2xy + 2y^2 = 1$,

$$x^2 - 2x\left(\frac{2}{3}x\right) + 2\left(\frac{2}{3}x\right)^2 = 1$$

$$\frac{5}{9}x^2 = 1$$

$$x^2 = \frac{9}{5}$$

$$x = \pm\sqrt{\frac{9}{5}}$$

So, $y = \frac{2}{3}\left(\pm\sqrt{\frac{9}{5}}\right) = \pm\frac{2}{\sqrt{5}} = \pm\frac{2}{5}\sqrt{5}$

Intersection of Line and Curve.

Equation of normal at $\left(\sqrt{\frac{9}{5}}, \frac{2}{5}\sqrt{5}\right)$:

$$y - \frac{2}{5}\sqrt{5} = 1\left(x - \sqrt{\frac{9}{5}}\right)$$

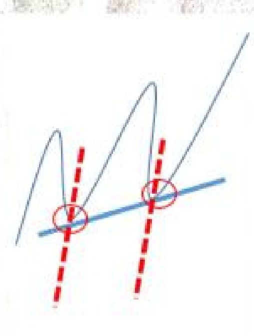
$$y = x + \frac{2}{5}\sqrt{5} - \frac{3}{5}\sqrt{5}$$

$$y = x - \frac{\sqrt{5}}{5}$$

or at $\left(-\sqrt{\frac{9}{5}}, -\frac{2}{5}\sqrt{5}\right)$

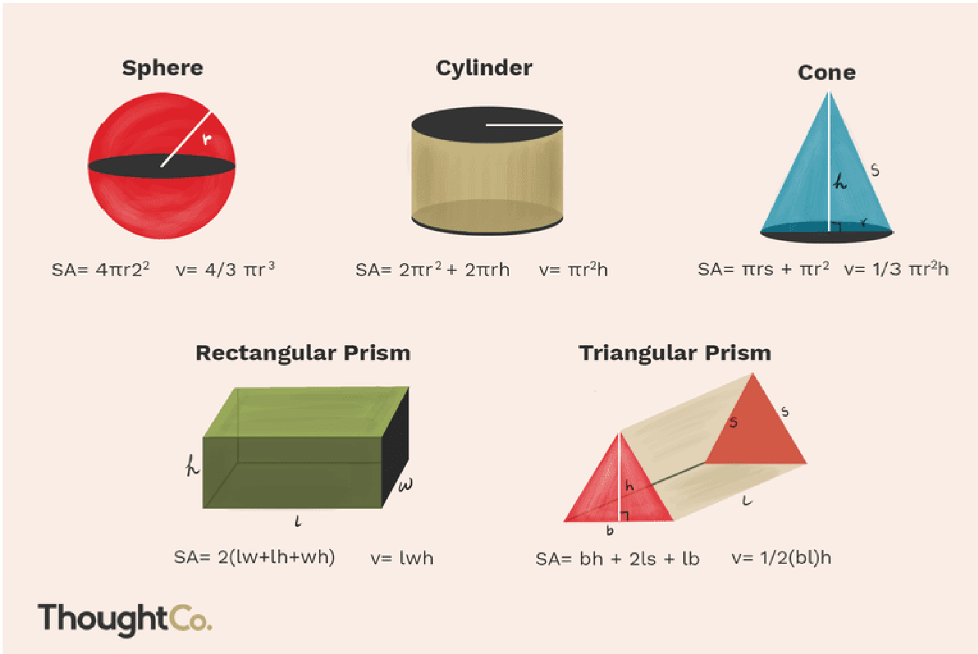
$$y + \frac{2}{5}\sqrt{5} = 1\left(x + \sqrt{\frac{9}{5}}\right)$$

$$y = x + \frac{\sqrt{5}}{5}$$



Source 7 (Khin 2019)

D. COMMON VOLUMES AND AREAS



E. DIFFERENTIATION USED TO FIND MAXIMUM AREA

Example

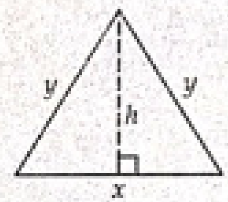
The perimeter of an isosceles triangle of base length x cm and two sides of equal length y cm each is 10 cm.
Show that its area can be expressed as $\frac{x}{2}\sqrt{25-5x}$.
Hence find its maximum area.

Solution

Perimeter of the isosceles triangle:

$$2y + x = 10 \Rightarrow y = 5 - \frac{x}{2}$$

$$h = \sqrt{y^2 - \left(\frac{1}{2}x\right)^2}$$
$$= \sqrt{\left(5-\frac{1}{2}x\right)^2 - \left(\frac{1}{2}x\right)^2}$$
$$= \sqrt{\left(5-\frac{1}{2}x-\frac{1}{2}x\right)\left(5-\frac{1}{2}x+\frac{1}{2}x\right)}$$
$$= \sqrt{5(5-x)}$$
$$= \sqrt{25-5x}$$



Area of triangle, $A = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times (x) \times h$$
$$= \frac{x}{2}\sqrt{25-5x}$$

$$\frac{dA}{dx} = \frac{1}{2}\sqrt{25-5x} + \frac{x}{2}\left(\frac{-5}{2\sqrt{25-5x}}\right)$$
$$= \frac{25-5x-\frac{5}{2}x}{2\sqrt{25-5x}}$$
$$= \frac{25-7.5x}{2\sqrt{25-5x}}$$

When $\frac{dA}{dx} = 0$

Maximum Area

$$25 - 7.5x = 0$$
$$x = 3\frac{1}{3} \text{ cm.}$$

Using sign text,

x	$3\frac{1}{3}^{-}$	$3\frac{1}{3}$	$3\frac{1}{3}^{+}$
$\frac{dA}{dx}$	+	0	−
	/	—	\

Hence area is max. when $x = 3\frac{1}{3}$ cm.

Since $2y + x = 10 \Rightarrow y = \dfrac{10 - 3\frac{1}{3}}{2} = 3\frac{1}{3}$.

Note: An equilateral triangle is formed.

$\therefore$ Area is a maximum when when $x = 3\frac{1}{3}$ cm and $y = 3\frac{1}{3}$ cm.

$$\text{Maximum area} = \frac{3\frac{1}{3}}{2}\sqrt{25 - 5\left(3\frac{1}{3}\right)} = 4.81\ \text{cm}^2.$$

Source 8 (Khin 2019)

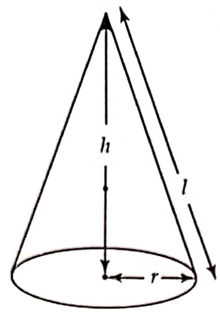
F. DIFFERENTIATION USED TO FIND MINIMUM AREA

Example

A conical shaped can of drink with base radius r, slant height of l and a vertical height h is made to hold a fixed volume V.

(a) Find an expression for the curved surface area, A, in terms of r, V and π.

(b) Find an expression for r in terms of V and π, which minimizes A.



Solution

(a) $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow h = \frac{3V}{\pi r^2}$

Since

$h^2 + r^2 = l^2$

$\Rightarrow l = \sqrt{h^2 + r^2}$

Hence,

$$l = \sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}$$

Now,

$$A = \pi r l$$

$$= \pi r\left(\sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}\right)$$

$$= \pi r\left(\sqrt{\frac{9V^2}{(\pi r^2)^2} + r^2}\right)$$

$$= \pi r\left(\sqrt{\frac{9V^2 + \pi^2 r^6}{(\pi r^2)^2}}\right)$$

$$= \frac{\sqrt{9V^2 + \pi^2 r^6}}{r}.$$

(ii) For min A, $\boxed{\frac{dA}{dr}} = \sqrt{9V^2+\pi^2r^6}\left(-\frac{1}{r^2}\right)+\left(\frac{1}{r}\right)\left(\frac{1}{2}\left(9V^2+\pi^2r^6\right)^{-\frac{1}{2}}\left(6\pi^2r^5\right)\right)\boxed{=0}$

Minimum Area

$$\frac{\sqrt{9V^2+\pi^2r^6}}{r^2}=\frac{3\pi^2r^4}{\sqrt{9V^2+\pi^2r^6}}$$

$$9V^2+\pi^2r^6=3\pi^2r^6$$

$$2\pi^2r^6=9V^2$$

$$r^6=\frac{9V^2}{2\pi^2}$$

$$r=\sqrt[6]{\frac{9V^2}{2\pi^2}}.$$

Source 9 (Khin 2019)

G. DIFFERENTIATION USED TO FIND RATES OF CHANGE OF VOLUME OR AREA

Problems can be solved by the following steps.

1. Identify what is to be found, e.g. $\frac{dA}{dt}$;
2. Identify what is given, e.g. $\frac{dx}{dt}$;
3. Using chain rule, find what is needed, e.g. $\frac{dA}{dt}=\boxed{\frac{dA}{dx}}\frac{dx}{dt}$;

   (In more complicated questions, we might need to apply chain rule twice, e.g. $\frac{dV}{dt}=\boxed{\frac{dV}{dA}\cdot\frac{dA}{dx}}\frac{dx}{dt}$.)
4. Find a relation between A and x; (Sometimes this is the difficult part)
5. Evaluate $\frac{dA}{dt}$.

Source 10 (Khin 2019)

Example

A spherical balloon is being inflated, and at the instant when its radius is 5 cm, its surface area is increasing at a rate of 2.8 $\text{cm}^2\ \text{s}^{-1}$. Find the rate of increase, at the same instant, of

(i) the radius,

(ii) the volume.

[The volume and the surface area of a sphere with radius r are respectively $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$.]

Solution

(i) $A = 4\pi r^2, \quad \frac{dA}{dt} = 2.8\ \text{cm}^3\text{s}^{-1}$ (given)

$$\frac{dA}{dr} = 8\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dA}\cdot\frac{dA}{dt}$$

$$= \frac{1}{8\pi r}\cdot(2.8)$$

Rate of change of radius

When $r = 5$,

$$\frac{dr}{dt} = \frac{1}{8\pi(5)}\cdot(2.8) = 0.0223\ \text{cm s}^{-1}$$

(ii) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$$

Rate of change of volume

$$= 4\pi r^2 \cdot \frac{1}{8\pi r}(2.8)$$

$$= \frac{5}{2}(2.8) \quad (\text{when } r = 5)$$

$$= 7\ \text{cm}^3\ \text{s}^{-1}.$$

Source 11 (Khin 2019)

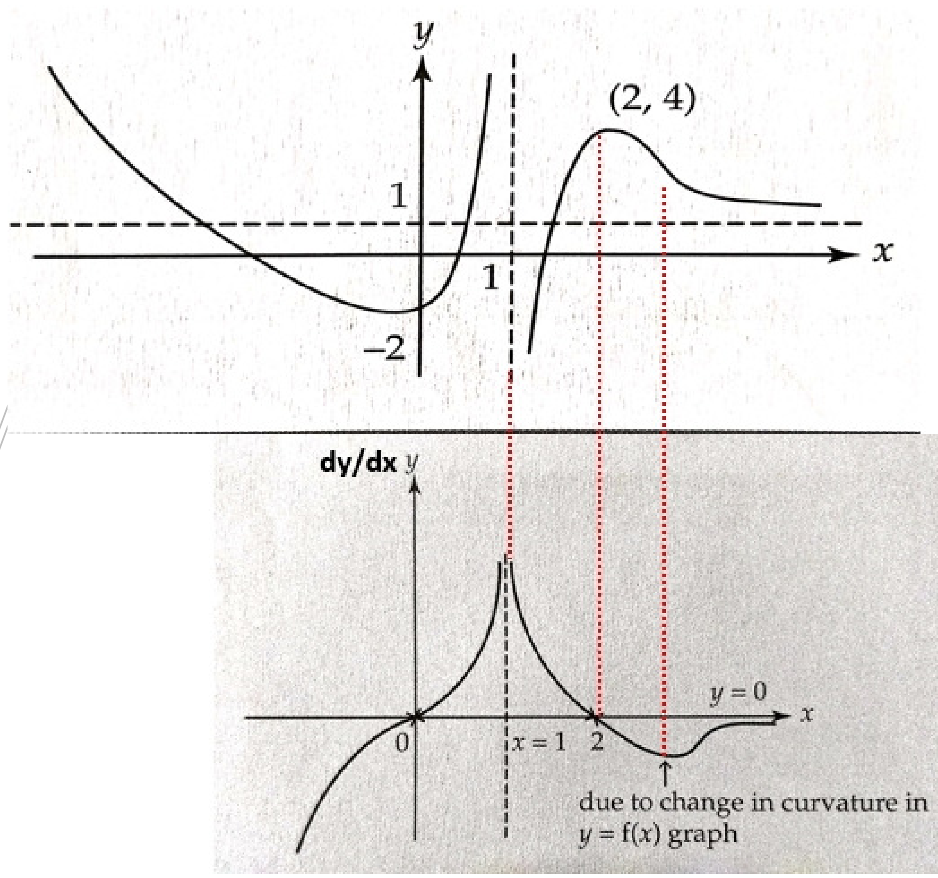
PART III

DERIVATIVES AND GRAPHS

A. FIRST DERIVATIVE

The purpose of the First Derivative $\frac{dy}{dx}$ is to get the Gradient Function.

B. GRAPH OF FIRST DERIVATIVE



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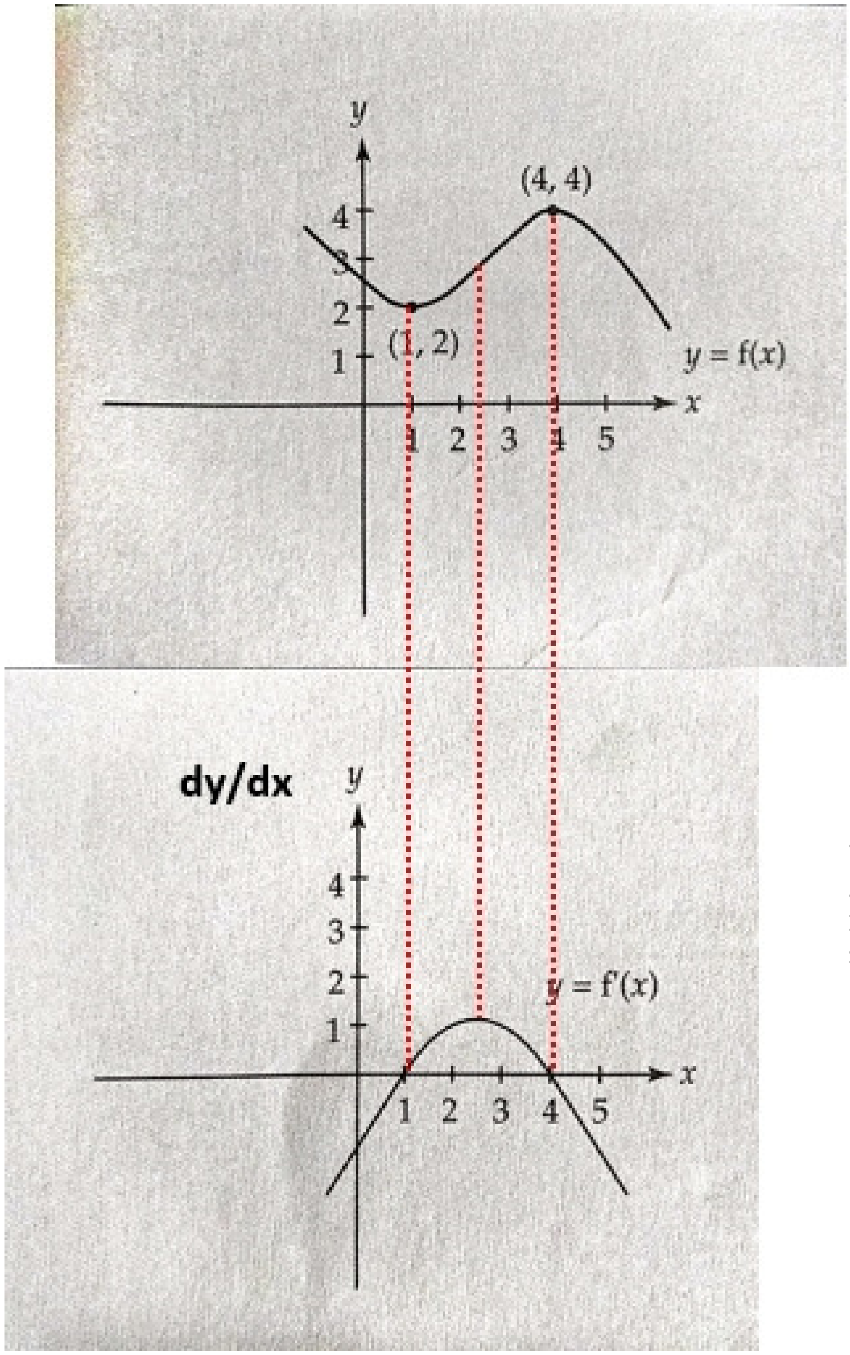

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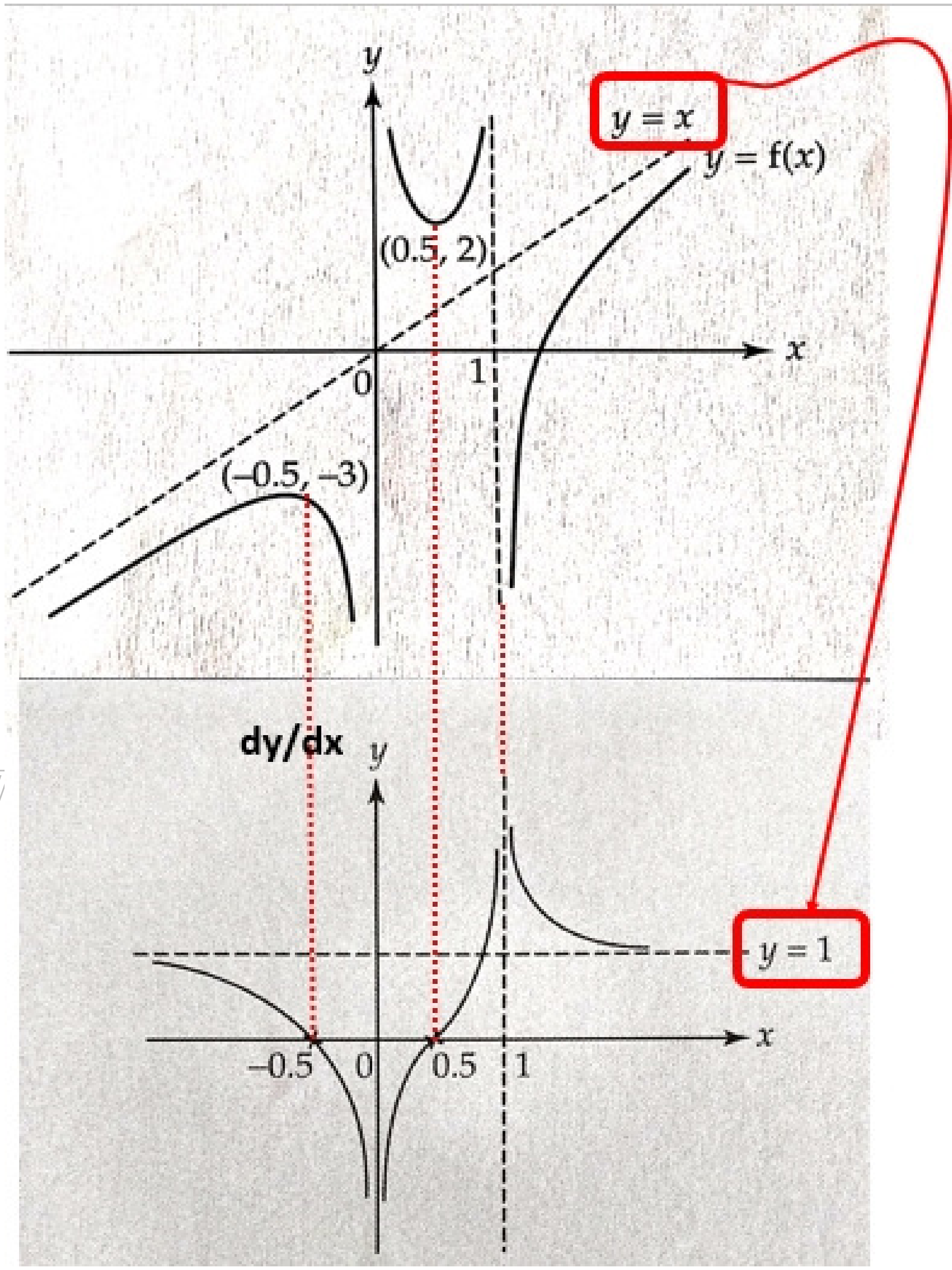

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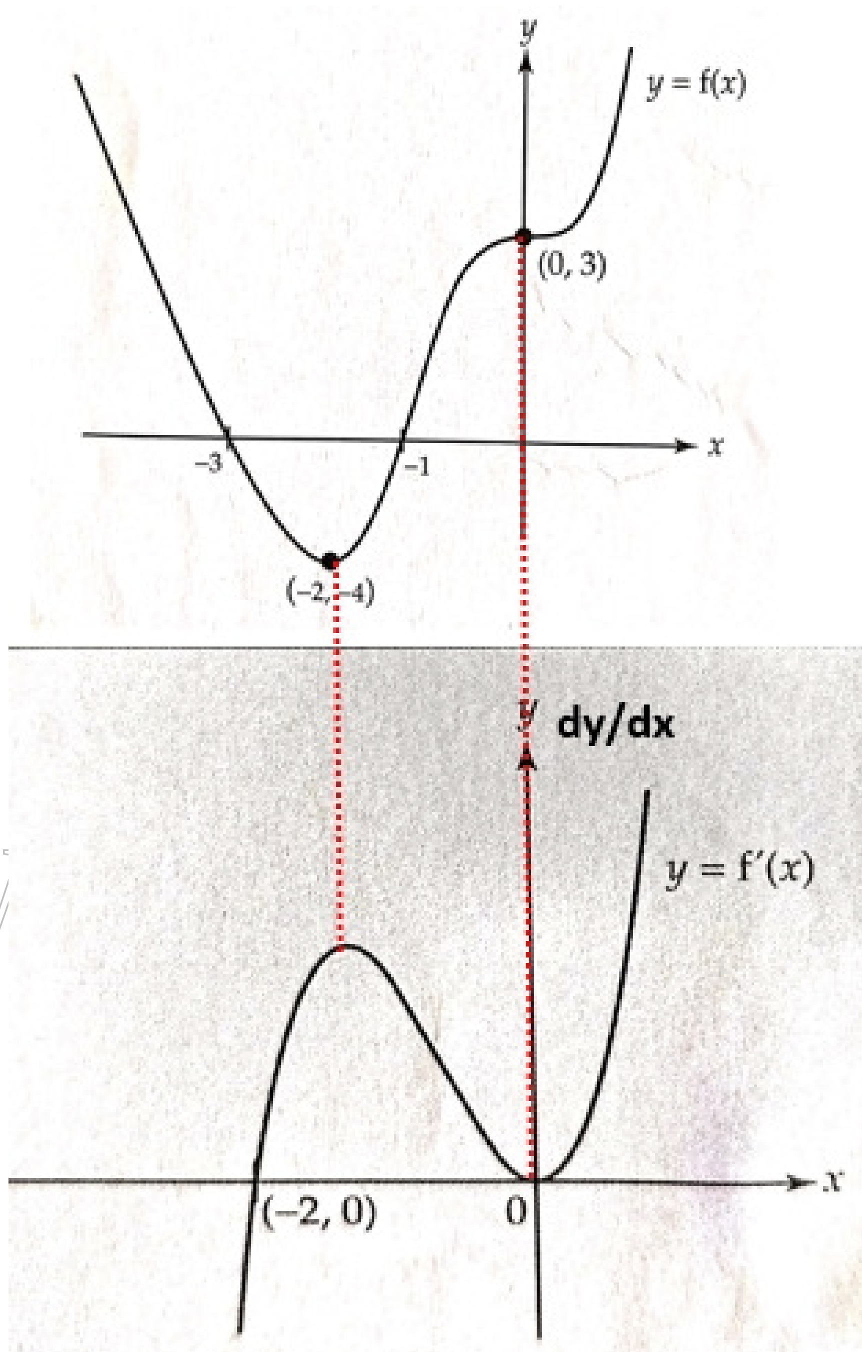
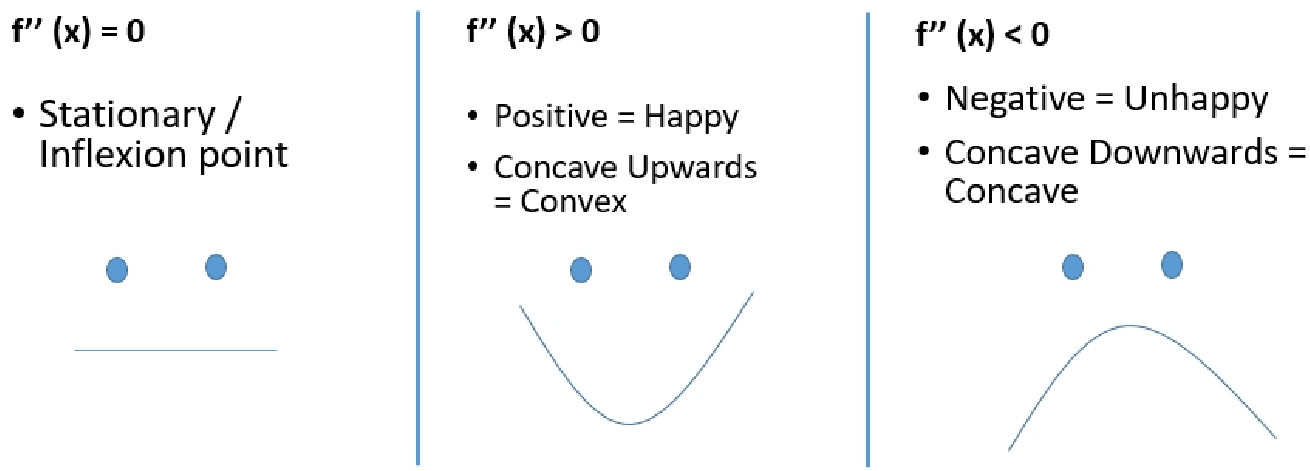


Source 15 (Khin 2019)

C. SECOND DERIVATIVE

- The purpose of the Second Derivative $\frac{d^2y}{dx^2}$ is to confirm if a stationary point is inflexion / Minimum / Maximum point.



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ABOUT MR SONG BOON KHING

Mr. Song Boon Khing graduated from NUS with a Bachelor of Science (2nd Upper Hons) degree, majoring in Applied Mathematics. Imbued with the passion to help and positively influence the young, Mr. Song applied and was awarded the MOE teaching award after graduating from Hwa Chong Junior College. Upon receiving his Post Graduate Diploma in Education (PGDE) with Credit, Mr. Song taught at National Junior College (NJC), teaching H1 and H2 A Level Mathematics.

ABOUT DR. ALVIN ANG

Dr. Alvin Ang earned his Ph.D., Masters and Bachelor degrees from NTU, Singapore. He is a scientist, entrepreneur, as well as a personal/business advisor. More about him at www.AlvinAng.sg.