DR. ALVIN'S PUBLICATIONS

LINEAR PROGRAMMING PART III

SENSITIVITY ANALYSIS

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CONTENTS

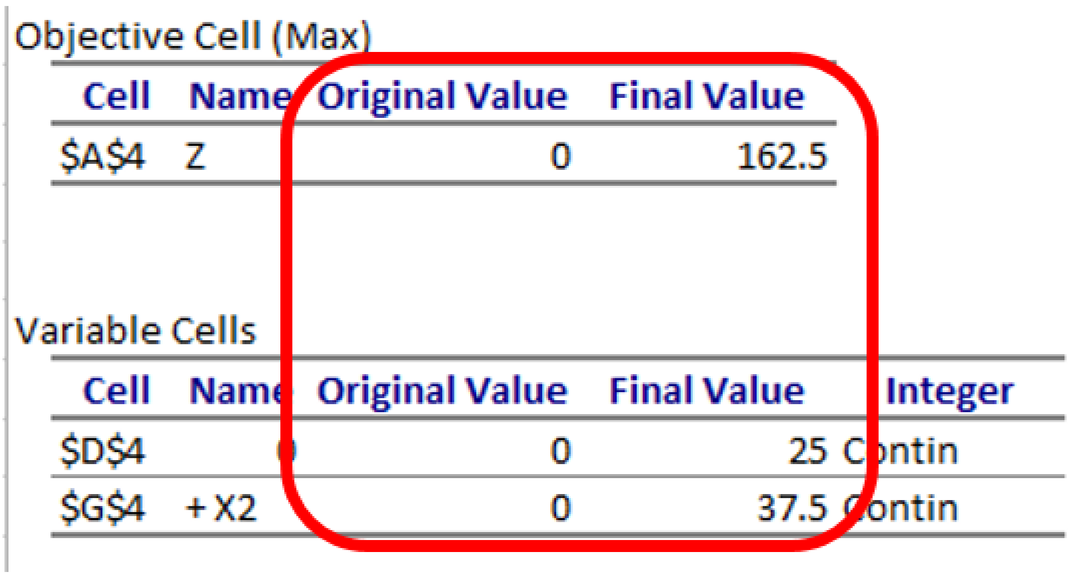
I. INTRODUCTION

This article follows after https://www.alvinang.sg/s/Linear-Programming-Part-II-Using-Excel-By-Dr-Alvin-Ang.pdf

Even though the preliminary results were shown in the previous section, we will have a deeper analysis of it here.

A. STEP 1: INTERPRETING THE ANSWER REPORT

The Answer Tab shows the Solver's results as shown below.



Objective Cell (Max)

Cell	Name	Original Value	Final Value
$A$4	Z	0	162.5

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$4		0	25	Contin
$G$4	+ X2	0	37.5	Contin

Figure 1: Solver's Results (Answer Report)

As can be seen in the "Final Value" Columns,

1. The maximum profit that Lucy's Madame can make at the fun fair selling her cupcakes is $162.50.
2. The optimal number of Vanilla cupcakes to bake is 25 while that for Chocolate cupcakes is 38.
3. The "Original Value" column next to the "Final Value" column simply states the values of the cells before Solver was run. In this case they are all zero because the cells were emptied before executing Solver. In other words, cells C8, C6 and D6 were empty before running Solver.

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C$18	Constraint 1 LHS	10	$C$18<=$D$18	Binding	0
$C$19	Constraint 2 LHS	5	$C$19<=$D$19	Not Binding	5
$C$20	Constraint 3 LHS	25	$C$20<=$D$20	Binding	0
$C$21	Constraint 4 LHS	37.5	$C$21<=$D$21	Not Binding	22.5

Figure 2: Solver's Results (Answer Report - Constraints)

The Answer Tab also shows the Constraints after solver was run.

1. WHAT DO THE CELL VALUES, BINDING AND SLACK COLUMN MEAN?

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$C\$18	Constraint 1 LHS	10	\$C\$18<=\$D\$18	Binding	0
\$C\$19	Constraint 2 LHS	5	\$C\$19<=\$D\$19	Not Binding	5
\$C\$20	Constraint 3 LHS	25	\$C\$20<=\$D\$20	Binding	0
\$C\$21	Constraint 4 LHS	37.5	\$C\$21<=\$D\$21	Not Binding	22.5

1. 1st Row – [Cell Value = 10 | Status = Binding | Slack = 0]
   - Recall Constraint 1: $(0.1)*(X_1) + (0.2)*(X_2) \leq 10$ kg
   - Recall Optimal Solution: ($X_1 = 25$, $X_2 = 37.5$)
   - Substitute Optimal Solution into Constraint 1: $(0.1)*(25) + (0.2)*(37.5) = 10$
   - This means that the 1st resource (Baking Powder) was fully utilized. All 10 kg was used up.
   - Since all 10kg was used, this inequality is bonded to 10 kg.
   - Hence, there is No Slack – meaning no extra resources for use anymore.

2. 2nd Row – [Cell Value = 5 | Status = Not Binding | Slack = 5]
   - Recall Constraint 2: $(0.05)*(X_1) + (0.1)*(X_2) \leq 10$ hours
   - Recall Optimal Solution: ($X_1 = 25$, $X_2 = 37.5$).
   - Substitute Optimal Solution into Constraint 2: $(0.05)*(25) + (0.1)*(37.5) = 5$
   - This means that the 2nd resource (man-hours) was NOT fully utilized. Only a total of 5 hours was used for baking (1.25 hours used for baking Vanilla Cupcakes and 3.75 hours used for baking Chocolate Cupcakes).
   - Since not all 10 hours was used, this inequality is not bonded to 10 hours.
   - Hence, there is a Slack of (10 – 5 = 5) hours – meaning an extra 5 hours can still be of use.
   - Meaning, Lucy's Madame can make use of this extra 5 hours to ask Lucy to do other household errands rather than wasting it away like chatting with other maids or playing with her hand phone.

3. 3rd Row – [Cell Value = 25 | Status = Binding | Slack = 0]
    - Recall Constraint 3: $X_1 \leq 25$ Vanilla cupcakes
    - Recall Optimal Solution: ($X_1 = 25$)
    - Substitute Optimal Solution into Constraint 3: $25 \leq 25$
    - This means that the 3rd constraint (maximum number of Vanilla Cupcakes) was fully utilized. All 25 Vanilla Cupcakes should be baked.
    - Since all 25 Vanilla Cupcakes should be baked, this inequality is bonded to 25.
    - Hence, there is No Slack – meaning no additional Vanilla Cupcakes can be baked anymore.

4. 4th Row – [Cell Value = 37.5 | Status = Not Binding | Slack = 22.5]
    - Recall Constraint 4: $X_2 \leq 60$ Chocolate cupcakes
    - Recall Optimal Solution: ($X_2 = 37.5$)
    - Substitute Optimal Solution into Constraint 4: $37.5 \leq 60$
    - This means that the 4th constraint (maximum number of Chocolate Cupcakes) was NOT fully utilized. Only 38 cupcakes should be baked.
    - Since only 38 Chocolate Cupcakes should be baked, this inequality is not bonded to 60.
    - Hence, there is a Slack of (60 – 37.5 = 22.5) Chocolate Cupcakes – meaning 22.5 Chocolate Cupcakes will not be baked.
    - The reason why an optimal solution of 38 Chocolate Cupcakes should be baked and not 60 is due to the following possibilities (in which the solver has already taken into account for all possibilities before giving the optimal answer)
        - ✓ Insufficient resources and/or
        - ✓ Does not maximize profits

B. STEP 2: INTERPRETING THE SENSITIVITY REPORT

(Aka. Sensitivity Analysis)

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$D$4	0	25	0	2	1E+30	0.5
$G$4	+X2	37.5	0	3	1	3

Figure 3: Sensitivity Analysis Report (Objective Function)

1. WHAT DO ALLOWABLE INCREASE AND DECREASE MEAN?

1. 1st Row – [Allowable Increase = 10^{30} (as good as ∞) | Allowable Decrease = 0.5]
   a. This means that I can increase the price of Vanilla Cupcake from $2 per cupcake to $∞ per cupcake, yet Optimal Solution: (X_1 = 25, X_2 = 37.5) will NOT change.
   b. Note that this may help increase profits, but if it is too expensive, nobody will buy.
   c. (The optimal number of vanilla cupcakes to be produced must still remain at 25).
   d. I can decrease the price of Vanilla Cupcake from $2 per cupcake to $(2 – 0.5 = 1.50) per cupcake, yet Optimal Solution: (X_1 = 25, X_2 = 37.5) will NOT change.

a) How is the Maximum Allowable Increase (10^{30}) and Decrease (0.5) Obtained?

i. Recall Objective Function: Max Z = **$2**$X_1$ + **$3**$X_2$

ii. Let C_1 : $2 → Max Z = **$$C_1$**$X_1$ + **$3**$X_2$

iii. Recall ***FIRST BINDING CONSTRAINT***:

- Constraint 1: (**0.1**)*(X_1) + (**0.2**)*(X_2) ≤ 10 kg

iv. Recall ***SECOND BINDING CONSTRAINT:***

- Constraint 3: **1***X_1 ≤ 25 Vanilla cupcakes

v. Using ***FIRST BINDING CONSTRAINT***:

- $\frac{\$C_1}{\$3} = \frac{0.1}{0.2} \rightarrow C_1 = \$1.5$
- This means that Max. Allowable Decrease for C_1 : \$2 → \$1.50
- Max. Allowable Decrease = \$0.50

vi. Using ***SECOND BINDING CONSTRAINT***:

- $\frac{\$C_1}{\$3} = \frac{1}{0} = \infty \rightarrow C_1 = \$\infty$
- This means that Max. Allowable Increase for C_1 : \$2 → \$∞
- Max. Allowable Increase = \$∞ = $\$10^{30}$

2. 2nd Row – [Allowable Increase = 1 | Allowable Decrease = 3]
   a. This means that I can increase the price of Chocolate Cupcake from \$3 per cupcake to \$4 per cupcake, yet Optimal Solution: (X_1 = 25, X_2 = 37.5) will NOT change.
      i. Once again, increasing the price may increase profits. But too expensive and people may not buy.
   b. I can decrease the price of Chocolate Cupcake from \$3 per cupcake to \$0 per cupcake, yet Optimal Solution: (X_1 = 25, X_2 = 37.5) will NOT change.

b) How is the Maximum Allowable Increase (1) and Decrease (3) Obtained?

i. Recall Objective Function: Max Z = **\$2**X_1 + **\$3**X_2

ii. Let C_2 : \$3 → Max Z = **\$2**X_1 + **\$$C_2$**$X_2$

iii. Recall ***FIRST BINDING CONSTRAINT***:

- Constraint 1: (**0.1**)*(X_1) + (**0.2**)*(X_2) ≤ 10 kg

iv. Recall ***SECOND BINDING CONSTRAINT:***

- Constraint 3: **<u>1</u>***$X_1 \leq 25$ Vanilla cupcakes

v. Using ***FIRST BINDING CONSTRAINT***:

- $\frac{\$2}{\$C_2} = \frac{0.1}{0.2} \rightarrow C_2 = \$4$
- This means that Max. Allowable Increase for C_2 : \$3 → \$4
- Max. Allowable Increase = \$1

vi. Using ***SECOND BINDING CONSTRAINT***:

- $\frac{\$2}{\$C_2} = \frac{1}{0} = \infty \rightarrow C_2 = \$0$
- This means that Max. Allowable Decrease for C_1 : \$3 → \$0
- Max. Allowable Decrease = \$3

2. 100% RULE FOR PRICING:

- Since we can easily increase the prices of the cupcakes (to raise profits) – without changing the optimal number of cupcakes baked, why don't we simply raise their prices to the maximum for **<u>BOTH</u>** cupcakes simultaneously?
- This is because it will infringe the 100% rule. In other words, we can increase / decrease the prices of the cupcakes only **<u>ONE</u>** at a time – in order to maintain the optimal number of cupcakes baked. Should we increase / decrease their prices simultaneously, it might infringe the 100% rule and the optimal numbers will shift (and we will need to re-run the solver).

3. WHAT IS THE 100% RULE?

For example, we would like to:

- Scenario 1:
    - Increase the price of each Vanilla cupcake by \$2 (to 2+2 = \$4) and
    - Increase the price of each Chocolate cupcake by \$2 (to 3+2 = \$5).
    - Is this possible?

        - No – because the maximum allowable increase for Chocolate cupcake is $1.
        - Should we wish to proceed to increase by $2 for Chocolate cupcake, the entire solver needs to be rerun.

- Scenario 2:
    - Increase the price of each Vanilla cupcake by $2 (to 2+2 = $4) and
    - Increase the price of each Chocolate cupcake by $1 (to 3+1 = $4).
    - Is this possible?
        - Check the 100% Rule.
        - $\sum \frac{Proposed\ Change}{Allowable\ Change} \leq 100$
        - Vanilla Cupcake: $\frac{Proposed\ Increase\ by\ \$2}{Maximum\ Allowable\ Increase} = \frac{\$2}{\$\infty} = 0$
        - Chocolate Cupcake: $\frac{Proposed\ Increase\ by\ \$1}{Maximum\ Allowable\ Increase} = \frac{\$1}{\$1} = 1$
        - 0 + 1 = 100% (≤100%) → Fits the 100% Rule
        - Therefore their prices can be increased without changing the Optimal Solution: ($X_1 = 25$, $X_2 = 37.5$).

- Scenario 3:
    - Decrease the price of each Vanilla cupcake by $0.50 (to 2 – 0.5 = $1.50) and
    - Decrease the price of each Chocolate cupcake by $1 (to 3 – 3 = $0).
    - Is this possible?
        - Check the 100% Rule.
        - $\sum \frac{Proposed\ Change}{Allowable\ Change} \leq 100$

- Vanilla Cupcake: $\frac{Proposed\ Decrease\ by\ \$0.50}{Maximum\ Allowable\ Increase} = \frac{\$0.50}{\$0.50} = 1$

- Chocolate Cupcake: $\frac{Proposed\ Decrease\ by\ \$1}{Maximum\ Allowable\ Decrease} = \frac{\$3}{\$3} = 1$

- 1 + 1 = 200% (≥100%) → Infringed the 100% Rule
- Therefore, if their prices are decreased, it will change the Optimal Solution: ($X_1 \neq 25$, $X_2 \neq 37.5$) → The Solver needs to be rerun.

4. **WHAT DOES REDUCED COST MEAN?**

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$D$4	0	25	0	2	1E+30	0.5
$G$4	+X2	37.5	0	3	1	3

- 1st & 2nd Row – [Reduced Costs = 0]
- Since both are $0, we cannot see its effect.
- They are both $0 because both X1 and X2 do contain a Final Value (25 and 37.5).
- To illustrate this, we will use an external example.

VARIABLE	VALUE	REDUCED COST
X1	3800.000	0.000
X2	0.000	0.250
X3	100.000	0.000
X4	3800.000	0.000
X5	300.000	0.000

Figure 4: External Example Answer Report

> x_2 is the number of seats purchased from DAP in quarter 1 and its current value is 0. Its reduced cost value is 0.25, indicating that if c_2 improved (decreased in this case) by 0.25 or more, x_2 would have a positive value in the new optimal solution. c_2 = $12.25 represents an improvement of exactly 0.25, so the answer is yes.

Figure 4 shows that the only reduced cost appears at the variable X2 (when its optimal value is 0). The coefficient (or cost) of X2 originally is $12.50. If this drops to below $12.25, then X2 will become ≥ 0 and no longer 0.

5. **WHAT DOES THE SHADOW PRICE MEAN?**

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$A$12	LHS	10	15	10	4.5	7.5
$A$13	LHS	5	0	10	1E+30	5
$A$14	LHS	25	0.5	25	75	25
$A$15	LHS	37.5	0	60	1E+30	22.5

Figure 5: Sensitivity Analysis Report (Constraints)

1. For Row 1, Shadow Price = $15 means that for every unit increase in Resource 1 (every additional 1 kg of Baking Powder given to Lucy), it will help increase Profits by $15 since now she has more Baking Powder to bake.

    a. Note that the maximum Baking Powder that can be given to Lucy is an additional 4.5 kg, while the maximum that can be taken away from her is 7.5 kg.

    b. If this range is exceeded, then the Optimal Solution will change and Solver needs to rerun.

2. For Row 2, Shadow Price = $0, which means that for every unit increase in Resource 2 (every additional 1 man-hour given to Lucy), it will not change Profits in any way.

3. For Row 3, Shadow Price = $0.50, which means that for every unit increase in Constraint 3 (every additional Forecasted Demand for Vanilla Cupcake), it will help increase Profits by $0.50.

    a. Note that the maximum forecasted demand for vanilla cupcake is an additional 75 cupcakes, while the maximum that can be reduced is by 25 cupcakes.

    b. If this range is exceeded, then the Optimal Solution will change and Solver needs to rerun.

For Row 4, Shadow Price = $0, which means that for every unit increase in Constraint 4 (every additional Forecasted Demand for Chocolate Cupcake), it will not change Profits in any way.

6. 100% RULE FOR CONSTRAINTS:

- Since we can easily increase the profits by giving Lucy more resources, why don't we simply raise **ALL** the resources to the maximum?
- This is because it will infringe the 100% rule. In other words, we can increase / decrease the resources given to Lucy only **ONE** at a time – in order to maintain the optimal number of cupcakes baked.
- Should we increase / decrease the resources simultaneously, it might infringe the 100% rule and the optimal numbers will shift (and we will need to re-run the solver).

7. WHAT IS THE 100% RULE?

For example, we would like to:

- Give Lucy additional 4.5 kg of Baking Powder and
- Increase the Forecasted Demand of Vanilla Cupcakes by 75.
    - Is this possible?
        - Check the 100% Rule.
        - $\sum \frac{Proposed\ Change}{Allowable\ Change} \leq 100\%$
        - Baking Powder: $\frac{Proposed\ Increase\ by\ 4.5kg}{Maximum\ Allowable\ Increase} = \frac{4.5kg}{4.5kg} = 1$
        - Vanilla Cupcake: $\frac{Proposed\ Increase\ by\ 75}{Maximum\ Allowable\ Increase} = \frac{75}{75} = 1$
        - 1 + 1 = 200% (≥100%) → Infringed the 100% Rule
        - Therefore, if the resources increased simultaneously, it will change the Optimal Solution: ($X_1 \neq 25$, $X_2 \neq 37.5$) → The Solver needs to be rerun.

REFERENCES

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ABOUT DR ALVIN ANG

Dr. Alvin Ang earned his Ph.D., Masters and Bachelor degrees from NTU, Singapore. He was a Professor as well as a personal/business advisor. More about him at www. AlvinAng.sg