DR. ALVIN'S PUBLICATIONS

VECTORS

DR. ALVIN ANG

CONTENTS

PART I

REPRESENTATION

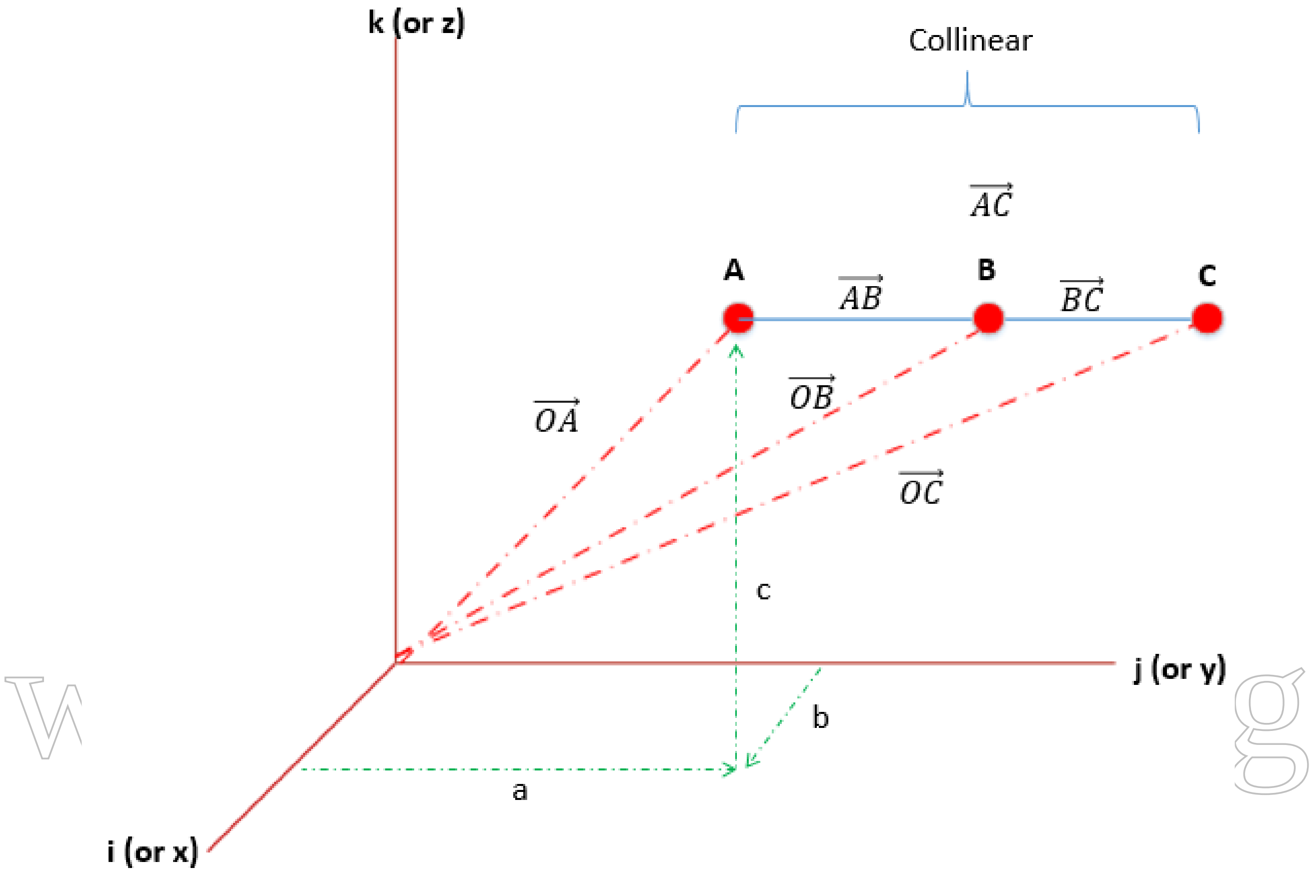


Figure 1: Representation of Vectors

A. ADDITION & SUBTRACTION

- $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB}$

B. LENGTH / MAGNITUDE

- Given: $\overrightarrow{OA} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\hat{i} + b\hat{j} + c\hat{k}$

- Magnitude of $\left|\overrightarrow{OA}\right| = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{a^2 + b^2 + c^2}$

C. PARALLEL

- AB is parallel to BC: $\overrightarrow{AB} // \overrightarrow{BC}$
- Then $\overrightarrow{AB} = k\overrightarrow{BC}$
- Where k is a scalar multiple.

D. COLLINEAR

- A, B and C are Collinear i.e. they lie on the same line.
- Therefore:
    - $\overrightarrow{AB} // \overrightarrow{BC}$
    - $\overrightarrow{AB} // \overrightarrow{AC}$
    - $\overrightarrow{BC} // \overrightarrow{AC}$

E. UNIT

- Given: $\overrightarrow{OA} = A = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\hat{i} + b\hat{j} + c\hat{k}$

- Unit Vector: $\hat{A} = \frac{A}{|A|} = \frac{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}{\sqrt{a^2 + b^2 + c^2}}$

F. DIRECTION RATIO & DIRECTION COSINES

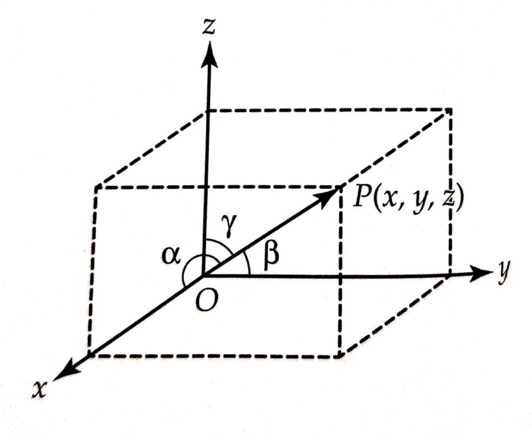


Figure 2: Direction Ratio and Direction Cosines (Khin 2019)

- Given: $\overrightarrow{OP} = r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- The Direction Ratio is x : y : z

- The Direction Cosines (l, m, n) are:

- $l = \cos\alpha = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$: α represents the angle r makes with the x axis.
- $m = \cos\beta = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$: β represents the angle r makes with the y axis.
- $n = \cos\gamma = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$: γ represents the angle r makes with the z axis.

G. RATIO THEOREM

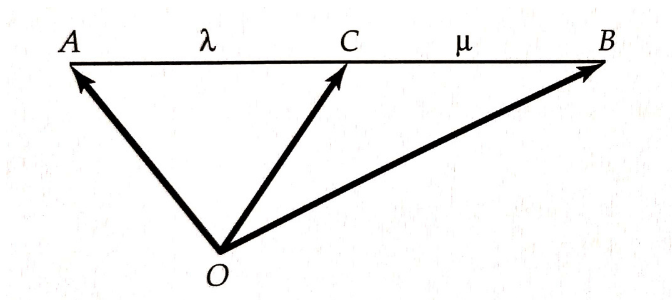


Figure 3: Ratio Theorem (Khin 2019)

- If AC : CB = λ : μ
- $\overrightarrow{OC} = \dfrac{\lambda\overrightarrow{OB} + \mu\overrightarrow{OA}}{\lambda + \mu}$

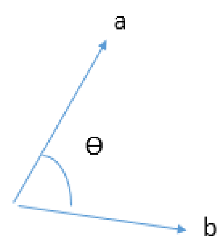
PART II

PRODUCT

A. SCALAR PRODUCT

DEFINITIONS

1) $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$



2) $\vec{a} \bullet \vec{b} = |a||b|\cos\theta$

PROPERTIES

1) $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$

2) $\vec{a} \bullet \left(\vec{b} + \vec{c}\right) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$

3) $\left(\vec{a} + \vec{b}\right) \bullet \left(\vec{c} + \vec{d}\right) = \left(\vec{a} \bullet \vec{c}\right) + \left(\vec{a} \bullet \vec{d}\right) + \left(\vec{b} \bullet \vec{c}\right) + \left(\vec{b} \bullet \vec{d}\right)$

4) $k\left(\vec{a} \bullet \vec{b}\right) = \left(k\vec{a}\right) \bullet \vec{b} = \vec{a} \bullet \left(k\vec{b}\right)$

- Where k is a scalar.

5) $\hat{a} \bullet \hat{a} = 1$

6) $\vec{a} \bullet \vec{a} = |a|^2$

APPLICATIONS

7) Angle between Vectors: $\theta = \cos^{-1} \frac{\vec{a} \bullet \vec{b}}{|a||b|}$

8) Perpendicular Vectors: If $\vec{a} \perp \vec{b}$, then $\vec{a} \bullet \vec{b} = 0$

9) Projection:

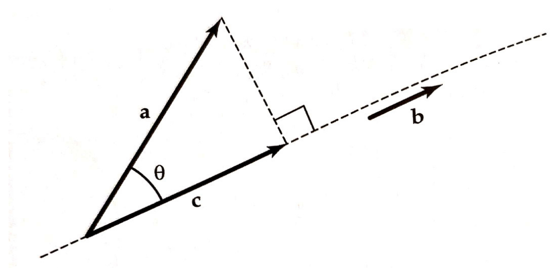


Figure 4: Projection of a Vector (Khin 2019)

- Length of Projection $|c| = \vec{a} \bullet \hat{b}$

- Where Vector $c = \left(\vec{a} \bullet \hat{b}\right)\hat{b}$

B. VECTOR CROSS PRODUCT

DEFINITIONS

1) $a \times b = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c$

2) Where:

- $c_1 = a_2 b_3 - a_3 b_2$
- $c_2 = -\left(a_1 b_3 - a_3 b_1\right)$
- $c_3 = a_1 b_2 - a_2 b_1$

3) Visual: (Right Hand Rule)

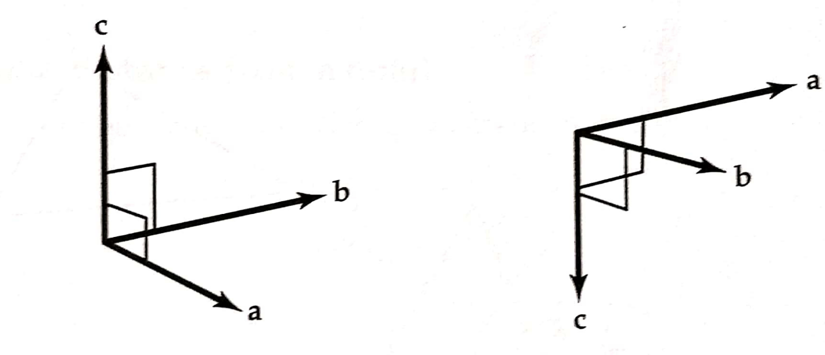


Figure 5: Cross Product (Khin 2019)

PROPERTIES

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2) $\left(k\vec{a}\right) \times \vec{b} = k\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \left(k\vec{b}\right)$

- Where k is a scalar.

3) $\vec{a} \times \left(\vec{b} + \vec{x}\right) = \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{x}\right)$

- Where $\vec{x}$ is some other vector.

4) $\left(\vec{a} + \vec{b}\right) \times \vec{x} = \left(\vec{a} \times \vec{x}\right) + \left(\vec{b} \times \vec{x}\right)$

5) $\vec{a} \bullet \left(\vec{b} \times \vec{x}\right) = \left(\vec{a} \times \vec{b}\right) \bullet \vec{x}$

6) $\vec{a} \times \left(\vec{b} \times \vec{x}\right) = \left(\vec{a} \bullet \vec{x}\right) \bullet \vec{b} - \left(\vec{a} \bullet \vec{b}\right) \bullet \vec{x}$

7) $\vec{a} \times \vec{a} = 0$

APPLICATIONS

1) $\vec{a} \times \vec{b} = n$

- Where n is the Normal of a Plane… discussed later..

2) Also, $\vec{a} \times \vec{b} = |a||b|\hat{n}\sin\theta$

- Where $\hat{n}$ is the perpendicular unit vector to Vectors **a** and **b**.

3) Area of Triangle

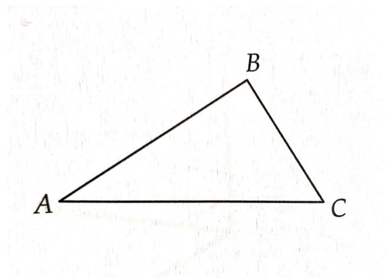


Figure 6: Area of a Triangle (Khin 2019)

- $$= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$
  $$= \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$$
  $$= \frac{1}{2}\left|\overrightarrow{CB} \times \overrightarrow{CA}\right|$$

4) Area of Parallelogram

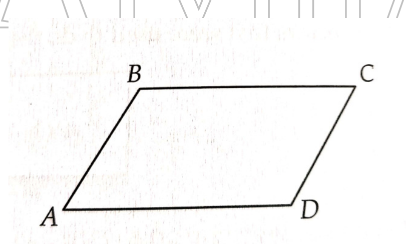


Figure 7: Area of Parallelogram (Khin 2019)

- $$= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$$
  $$= \left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$$
  $$= \left|\overrightarrow{CB} \times \overrightarrow{CD}\right|$$
  $$= \left|\overrightarrow{DC} \times \overrightarrow{DA}\right|$$

5) Perpendicular Distance from a Point

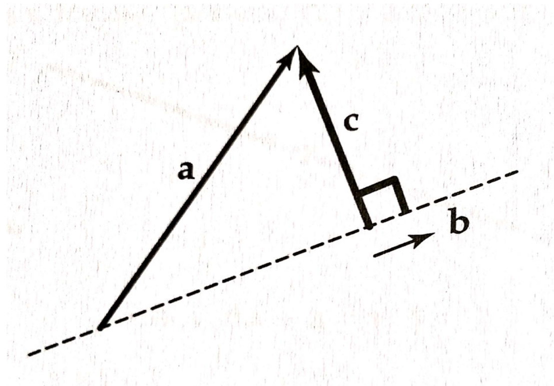


Figure 8: Perpendicular Distance from a Point (Khin 2019)

- $|c| = |a \times \hat{b}|$

PART III

LINES

A. VECTOR EQUATION OF A LINE

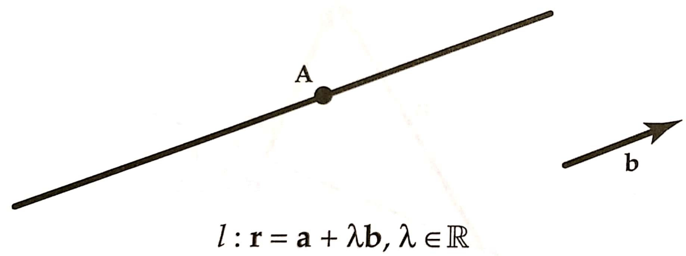


Figure 9: Vector Equation of a Line (Khin 2019)

- $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \vec{a}$

- $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \vec{b}$

- If line l passes thru ***a*** and is parallel to ***b***,
- Then the General Equation of the line l is $r = \vec{a} + \lambda\vec{b}$
- And if a point P lies on the line l,
- Then $\overrightarrow{OP} = \vec{a} + \lambda\vec{b}$
- Note: The difference between $\overrightarrow{OP}$ and r is that r is the General Equation (λ remains as a variable).
- But with a fixed λ, the point P is determined.

B. CONVERTING VECTOR EQUATION INTO CARTESIAN EQUATION

- Since the equation of a line is $r = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- The Cartesian Equation is $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$

C. POINT & LINE

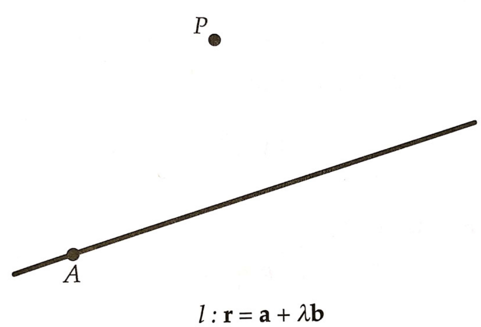


Figure 10: Point & Line (Khin 2019)

D. DISTANCE BETWEEN POINT & LINE

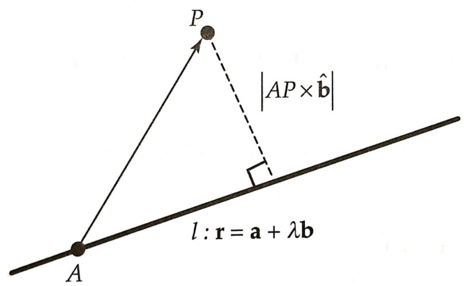


Figure 11: Distance between Point and Line (Khin 2019)

- The Distance between a Point and a Line is $\left|\overrightarrow{AP} \times \hat{b}\right|$

E. FOOT OF A PERPENDICULAR

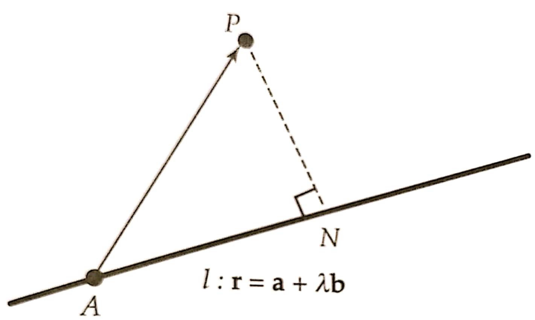


Figure 12: Foot of a Perpendicular (Khin 2019)

- Finding the Foot of a Perpendicular:
  - ✓ Step 1: Let N be the Foot of the Perpendicular
  - ✓ Step 2: Since N lies on the line l, $\overrightarrow{ON} = a + \lambda b$ (for some value of λ)
  - ✓ Step 3: Since $\overrightarrow{PN} \perp l$, thus $\overrightarrow{PN}.b = 0$
  - ✓ Step 4: From Step 3, we can find λ, and we can find $\overrightarrow{ON}$

F. REFLECTION OF POINT ABOUT LINE

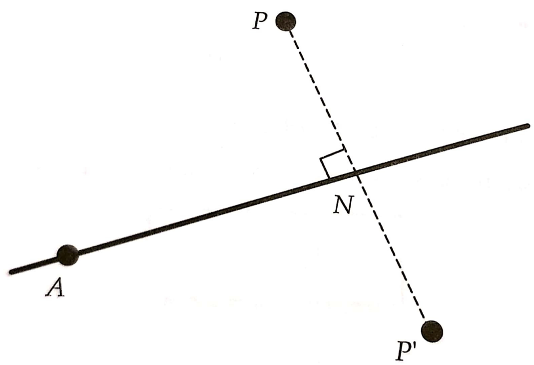


Figure 13: Reflection of Point about Line (Khin 2019)

- Finding $\overrightarrow{OP'}$:
- $\overrightarrow{ON} = \dfrac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$
- $\overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP}$

G. PARALLEL LINES

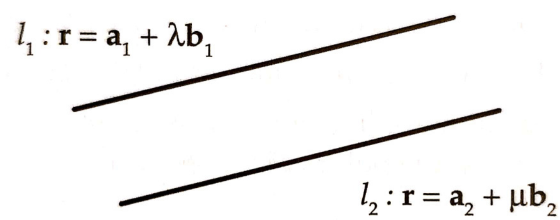


Figure 14: Parallel Lines (Khin 2019)

- If l_1 and l_2 are parallel, then $b_1 = kb_2$

H. DISTANCE BETWEEN TWO PARALLEL LINES

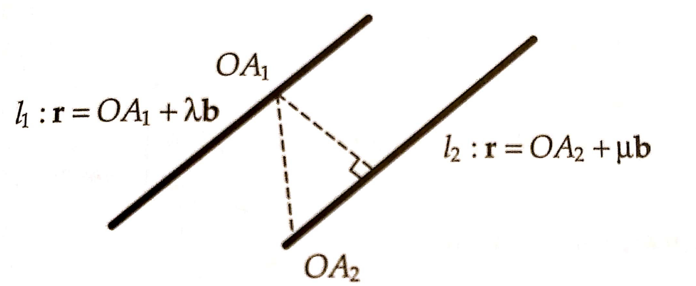


Figure 15: Distance between Two Parallel Lines (Khin 2019)

- Distance between Two Parallel Lines = $\left|\overrightarrow{A_1A_2} \times \hat{b}\right|$

I. INTERSECTING LINES

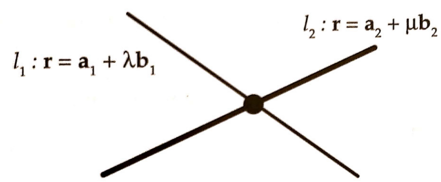


Figure 16: Intersecting Lines (Khin 2019)

- How to check if l_1 and l_2 intersect?
- Let
  - ✓ $a_{11} + \lambda b_{11} = a_{21} + \mu b_{21}$
  - ✓ $a_{12} + \lambda b_{12} = a_{22} + \mu b_{22}$
  - ✓ $a_{13} + \lambda b_{13} = a_{23} + \mu b_{23}$
- Solve any 2 of the 3 equations above to obtain λ & μ
- Once λ & μ is obtained, if all equations above show LHS = RHS → Means l_1 and l_2 intersect.

- Else, they do not intersect.

J. SKEW LINES

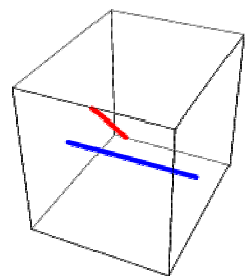


Figure 17: Skew Lines

- In three-dimensional geometry, skew lines are two lines that do not intersect and are not parallel.

K. ANGLE BETWEEN TWO LINES

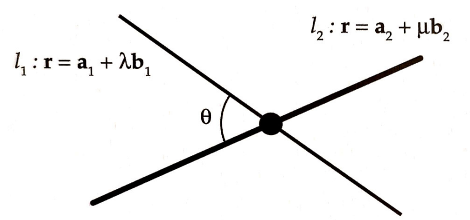


Figure 18: Angle between Two Lines (Khin 2019)

- To find angle between two lines: $\cos\theta = \left|\hat{b}_1 . \hat{b}_2\right|$

L. REFLECTION OF A LINE

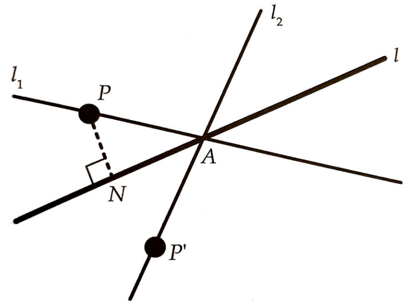


Figure 19: General Equation for Hyperbolas (Khin 2019)

- Presume l_1 is mirror reflected about l to form l_2.
- How do we find l_2?
- Step 1: Find $\overrightarrow{OA}$ (the intersection between l_1 and l)
  - ✓ Refer to Figure 16: Intersecting Lines (Khin 2019) to find intersection.
- Step 2: Obtain $\overrightarrow{OP}$ (any point lying on the line l_1)
  - ✓ $\overrightarrow{OP}$ could also possibly be $\overrightarrow{OP} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}$
  - ✓ Where $\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}$ is gotten from $l_1 : r = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} + \lambda \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix}$
- Step 3: Find $\overrightarrow{ON}$ (foot of the perpendicular)
  - ✓ Refer to Figure 12: Foot of a Perpendicular (Khin 2019) to find the foot of a perpendicular.
- Step 4: Find $\overrightarrow{OP'}$
  - ✓ Refer to Figure 13: Reflection of Point about Line (Khin 2019) to find $\overrightarrow{ON} = \frac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$ → $\overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP}$
- Step 5: Obtain l_2
  - ✓ $l_2 : \ r = \overrightarrow{OP'} + \lambda \overrightarrow{AP'}$

PART IV

PLANES

A. PARAMETRIC FORM

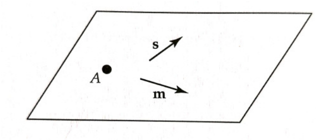


Figure 20: Plane in Parametric Form (Khin 2019)

- A: a Point on the plane
- s: a free vector parallel to / on the plane
- m: another free vector parallel to / on the plane (but not parallel to s)
- The plane $\Pi: r = a + \lambda m + \mu s$

B. SCALAR PRODUCT FORM

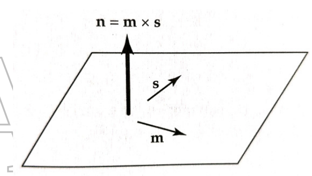


Figure 21: Normal to the plane (Khin 2019)

- $\Pi: r.n = d$
- Where $n = m \times s$
- Where $d = a.n$

C. CARTESIAN FORM

- Since the Scalar form is $r.\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d$, the Cartesian form becomes $n_1x + n_2y + n_3z = d$.

D. POINT AND PLANE

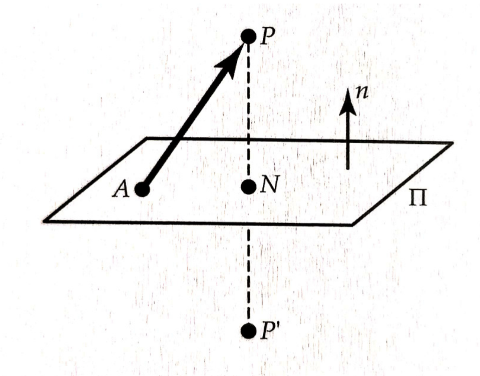


Figure 22: Point and Plane (Khin 2019)

- Referring to Figure 4: Projection of a Vector (Khin 2019), we make use of it to obtain the ***Distance from P to N*** → $\left|\overrightarrow{PN}\right| = \left|\overrightarrow{AP}.\hat{n}\right|$

- Still using Figure 4: Projection of a Vector (Khin 2019), we make use of it to obtain the ***Foot of the Perpendicular, N***.

  - $\overrightarrow{PN} = \overrightarrow{PO} + \overrightarrow{ON} = \left(\overrightarrow{PA}.\hat{n}\right)\hat{n}$

  - Thus $\overrightarrow{ON} = \overrightarrow{OP} + \left(\overrightarrow{PA}.\hat{n}\right)\hat{n}$

- Since $\overrightarrow{PN} = \overrightarrow{NP'}$, we can find the ***Point of Reflection***, $\overrightarrow{OP'}$

E. PARALLEL PLANE AND LINE

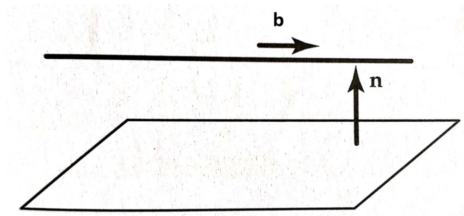


Figure 23: Parallel Plane and Line (Khin 2019)

- Let line $l : r = a + \lambda b$
- Let plane $\Pi : r.n = d$
- If l and Π are parallel, then $n.b = 0$

F. LINE LIES ON THE PLANE

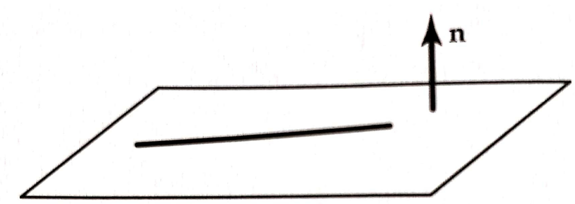


Figure 24: Line lies on the Plane (Khin 2019)

- Let line $l : r = a + \lambda b$
- Let plane $\Pi : r.n = d$
- If line l lies on plane Π, then both "r" are equal → $(a + \lambda b).n = d$ (for all values of λ, meaning, that λ is a variable here, not a constant).

G. LINE INTERSECTS THE PLANE

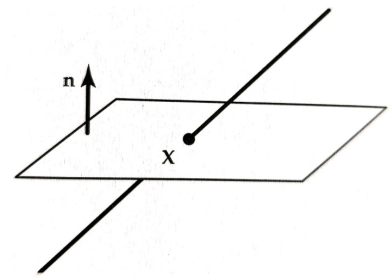


Figure 25: Line Intersects the Plane (Khin 2019)

- Let line $l: r = a + \lambda b$
- Let plane $\Pi : r.n = d$
- If line l is not parallel to plane Π, then it must intersect at X.
- Once again, both "r" are equal → $(a + \lambda b).n = d$
- But this time, λ can be solved to a single constant value.
- Thus $\overrightarrow{OX}$ can be found.

H. FOOT OF PERPENDICULAR FROM POINT TO PLANE

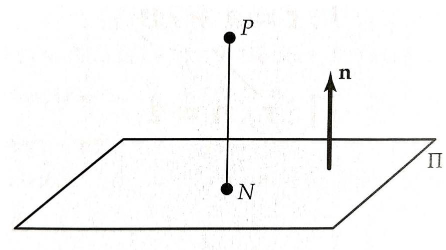


Figure 26: Foot of Perpendicular from Point to Plane (Khin 2019)

- Let line PN $l_{PN} : r = p + \lambda n$
- Let plane $\Pi : r.n = d$

- Equate both "r" → $(p+\lambda b).n = d$ to solve for λ

- Substitute λ back into l_{PN} to obtain $\overrightarrow{ON}$ (the point of intersection / foot of perpendicular)

I. REFLECTION OF LINE ABOUT PLANE

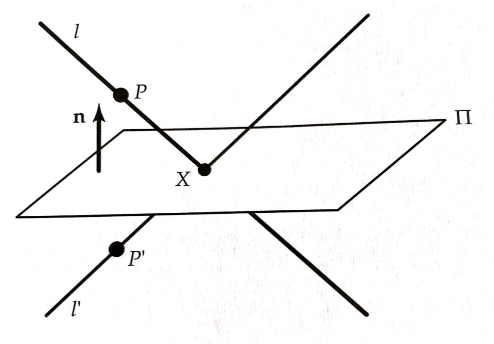


Figure 27: Reflection of Line about Plane (Khin 2019)

- Let line $l : r = a + \lambda b$

- Let plane $\Pi : r.n = d$

- Step 1: Refer to Figure 25: Line Intersects the Plane (Khin 2019) to obtain $\overrightarrow{OX}$

- Step 2: Refer to Figure 22: Point and Plane (Khin 2019) to obtain $\overrightarrow{OP'}$

- Step 3: Use P' and X to form the line l'.

J. ANGLE BETWEEN LINE AND PLANE

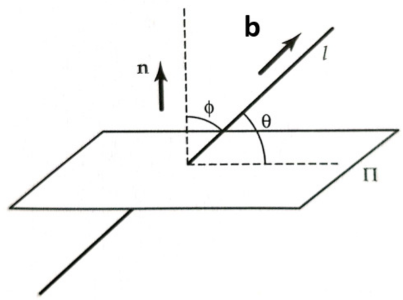


Figure 28: Angle between Line and Plane (Khin 2019)

- Let line $l : r = a + \lambda b$
- Let plane $\Pi : r.n = d$
- To find $\phi \rightarrow \cos\phi = |\hat{n}.\hat{b}|$
- To find $\theta \rightarrow \sin\theta = |\hat{n}.\hat{\mathrm{b}}|$

K. PARALLEL PLANES

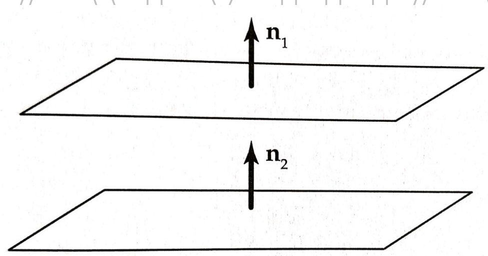


Figure 29: Parallel Planes (Khin 2019)

- Let plane $\Pi_1 : r.n_1 = d_1$
- Let plane $\Pi_2 : r.n_2 = d_2$
- If $\Pi_1 // \Pi_2$, then $n_1 = kn_2$

- Furthermore, if $n_1 = n_2$, then the Perpendicular Distance between Π_1 & Π_2 is $\left|\frac{d_1 - d_2}{|n|}\right|$

L. INTERSECTING PLANES

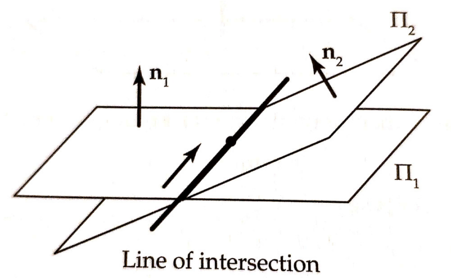


Figure 30: Intersecting Planes (Khin 2019)

- How to find the intersection of planes Π_1 and Π_2 i.e. line l?
- Let line $l : r = a + \lambda b$
- We find b by $n_1 \times n_2$ (since $l // n_1 \times n_2$)
- Then we find a by resolving 2 equations with 2 unknowns.
- This is done by:
  - Let $r = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} or \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} or \begin{pmatrix} o \\ y \\ z \end{pmatrix}$
  - Then sub this r into Π_1 and Π_2 to get 2 equations 2 unknowns.
  - You can refer to the section of "How to Solve 2 equations 2 unknowns" using TI 84 calculator by Ang (2019).

M. ANGLE BETWEEN PLANES

$$\cos\theta = \left|\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2\right|.$$

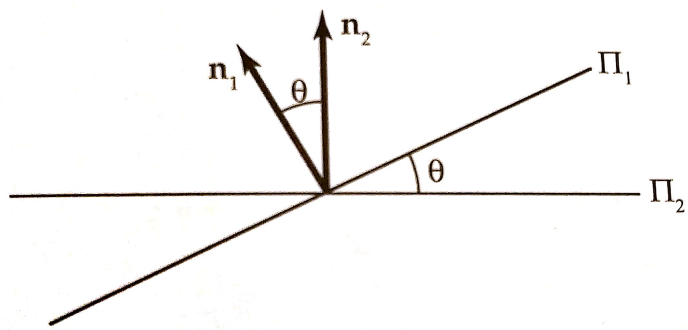


Figure 31: Angle Between Planes (Khin 2019)

- To find the angle between planes Π_1 and Π_2: $\cos\theta = |\hat{n}_1 . \hat{n}_2|$

REFERENCES

Ang, A. (2019). Some Useful TI 84 GC Functions.

Khin, S. B. (2019). Effective Guide (H2) Mathematics, Fairfield Book Publishers.

ABOUT MR SONG BOON KHING

Mr. Song Boon Khing graduated from NUS with a Bachelor of Science (2nd Upper Hons) degree, majoring in Applied Mathematics. Imbued with the passion to help and positively influence the young, Mr. Song applied and was awarded the MOE teaching award after graduating from Hwa Chong Junior College. Upon receiving his Post Graduate Diploma in Education (PGDE) with Credit, Mr. Song taught at National Junior College (NJC), teaching H1 and H2 A Level Mathematics.

ABOUT DR. ALVIN ANG

Dr. Alvin Ang earned his Ph.D., Masters and Bachelor degrees from NTU, Singapore. He is a scientist, entrepreneur, as well as a personal/business advisor. More about him at www.AlvinAng.sg.